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Statistical Properties of a Nonuniformly Hyperbolic Map of the Interval

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We prove a power-law upper bound for the decay of the correlations for Hölder observables in the case of a nonuniformly hyperbolic map of the interval introduced by Gaspard and Wang as a piecewise linear approximation of the intermittent map of Manneville–Pomeau. The result is then applied to compute the Central Limit Theorem for the same class of observables.

KEY WORDS: Decay of correlations; Central Limit Theorem; symbolic dynamics; Markov chains.

1. INTRODUCTION

We compute the decay of the correlations and the Central Limit Theorem (CLT) for Hölder continuous functions in the case of the 1D nonuniformly hyperbolic map of the unit interval introduced by Gaspard and Wang[1,2] as an approximation of the Manneville-Pomeau intermittent system[3]; see also ref. 7 for recent studies on this map, in particular, on the occurrence time of long laminar periods. As pointed out in ref. 1, this map exhibits long-range temporal correlations; in fact, it admits the origin 0 as an indifferent fixed point and whenever an orbit enters the neighborhood of the origin, then it will take a long time before getting out. This kind of behavior is paradigmatic of other, more complex, dynamical systems, where the nonuniform hyperbolicity is responsible for the slow decay of correlations (usually of power type). Among these systems we recall the continuous-time flow in the Lorentz gas without horizon, for which the

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correlations are conjecture to be as $1/T$, with T the time; the Bunimovich stadium, for which in ref. 14 it conjectured a power-law decay of correlations; and finally a dispersing billiard on a table bounded by three mutually tangent arcs, for which ref. 15 formulated a similar conjecture.

It is known that, depending upon a structure parameter α, the invariant measure of our transformation can be finite or not: the former case is realized for $\alpha > 1$. We rigorously establish a power-law decay of the correlations when $\alpha > 3$, but we guess that it continues to persist when $1 < \alpha \leqslant 3$; see Section 5 for a critical discussion of this point. The CLT can be rigorously verified for Hölder observables whenever $\alpha > 30$, although we think it holds for $2 < \alpha \leqslant 30$, too. The nature of all these bounds for the parameter α is due to the method we used, when is an application of symbolic dynamics on topological denumerable Markov chains and is reminiscent of the technique of Ibragimov type used in ref. 4 to compute the decay of correlations for dispersing billiards. To better understand this point, we now outline the idea of the proof. Starting with a denumerable Markov partition of the interval $[0, 1]$, we construct a Markov sieve $\mathcal{M}_{n,n_1,n_2}$ (we adopt the terminology of ref. 4), which is another partition of the unit interval depending on the three indices: n is the order of iteration entering the correlation function, $n_1 < n$ is the length of the cylinders on which we approximate our observables with piecewise constant ones, and finally $n_2 < n_1$ gives a bound to the infinite alphabet of the associated Markov chain. In particular, $\mathcal{M}_{n,n_1,n_2} = \hat{\Omega}_{n,n_1,n_2} \cup \tilde{\Omega}_{n,n_1,n_2}$, where $\hat{\Omega}_{n,n_1,n_2}$ will be discarded while the statistical analysis will be performed on $\tilde{\Omega}_{n,n_1,n_2}$, which we can endow with a structure of nonstationary finite Markov chain (after having renormalized the original measure on it). The dominant term of the decay comes from the measure of $\hat{\Omega}_{n,n_1,n_2}$ and, due to the nonuniform hyperbolicity of the mapping, is of the form n/n_2^{α}. This obliges us to choose $n_2 = n^{\eta}$, with $\eta < 1$ and $\eta > 1/\alpha$. Conditions of the same type are necessary to ensure a subexponential decay of the errors arising either by approximating the original observable with cylindrical functions of length n_1 and the stationary conditional probabilities with the corresponding nonstationary ones. In particular they make the preceding estimate worse, giving $\eta > 1/(\alpha - 1)$.

Finally, the statistical analysis on $\tilde{\Omega}_{n,n_1,n_2}$ will produce a subexponential bound provided that $\eta < 1/2$. This explains why we have to keep $\alpha > 3$. More stringent conditions of the same nature are met in the proof of the CLT, requiring $\alpha > 30$. We want to point out that also for dispersing billiards there is a discarded set, close to the singularity lines (expressed by the "rank condition"[5]), but due to the uniform hyperbolicity of the system, the measure of this set is exponentially small, so that the leading term for the correlation is given by the subexponential decay arising from the finite nonstationary Markov chain.

We recall that in our case a power-law decay was already obtained by Wang[1] for the autocorrelation of the characteristic function of the "chaotic" region Δ_0 (see Section 2), but his argument is not conclusive. However, for such an observable the correlations can be written quite easily and computed numerically without approximations of any sort and reveal just a decay of power type.[5] This confirms the validity of our analysis and suggests the impossibility of getting a decay faster than the power one, at least in the class of Hölder observables. We will also show that the subexponential rate of decay of the nonstationary (finite) Markov chain (Lemma 3.8) is confirmed by an extremely accurate numerical analysis. This analysis, shown in Section 5, verifies a sort of Döeblin condition for the convergence of the ratio of the conditional probabilities. Also in Section 5 we show that the theoretical estimation of the measure of $\hat{\Omega}_{n,n_1,n_2}$ is confirmed by the numerical computations. In our opinion a reliable and accurate numerical analysis of the correlations is very important just to find the real natural of the decay compared with the theoretical predictions, which are often worse. However, it is a matter of fact that for nonuniformly hyperbolic systems or for systems with singularities, such numerical computations can be done only for a small order of iteration n; as a consequence, the statistics is so poor as to prevent any decisive conclusion. These questions are well illustrated in the statistical analysis of dispersing billiards, where, along with the theoretical estimations, there is an increasing list of numerical works giving qualitatively different results for the decay of correlations (see the Introduction of ref. 13 for a discussion of this point and related references). Therefore it seems interesting to us that, for our system, we are able to work out some quantities of statistical importance with great precision and accuracy.

As a final comment, we observe that besides the symbolic dynamics, the other powerful technique to study the statistical properties of dynamical systems is given by the Perron–Fröbenius theory (see, for example, refs. 8–12 for the analysis of 1D mappings). It would be interesting to investigate if such a method could get and improve our results to $1 < \alpha \leqslant 3$.

2. THE SYSTEM AND STATEMENTS OF THE RESULTS

2.1. We now recall the definition and the main properties of the Gaspard–Wang mapping[1,2]; it is a piecewise linear mapping of the unit interval onto itself defined as

[5] In particular, for the autocorrelation of the characteristic function of Δ_0 the numerical analysis gives a decay of type $n^{-(\alpha-1)}$, while we found in this paper an analytic (asymptotic) bound like $n^{-(\alpha/2-1)}$.

$$x_{n+1} = f(x_n) = \begin{cases} \dfrac{\xi_{k-2} - \xi_{k-1}}{\xi_{k-1} - \xi_k}(x_n - \xi_k) + \xi_{k-1} & \text{if} \quad \xi_k \leqslant x_n < \xi_{k-1} \\ \dfrac{x_n - a}{1-a} & \text{if} \quad a \leqslant x_n < 1 \end{cases}$$

with $\xi_k = a/(1+k)^\alpha$, $k \in \mathbf{Z}^+$, $\xi_0 = a$, $\xi_{-1} = 1$ (see Fig. 1).

We choose $\alpha > 1$; under this assumption it is possible to prove the existence of a finite invariant probability measure μ. This is a Markov measure. To understand the properties of such a measure, we begin by observing that the partition (mod 0) of $[0, 1]$ into the intervals $\Delta_i = \xi_{i-1} - \xi_i$, $i \geqslant 0$, is a Markov partition. Denote by Ω the space of one-sided sequences $\omega \equiv (\omega_0, \omega_1, ..., \omega_n, ...)$, $\omega_i \in \{0, 1, 2, ...\}$, satisfying the compatibility condition: given ω_i, then $\omega_{i-1} = \omega_i + 1$ or $\omega_{i-1} = 0$, and the map ϕ associating to $x \in [0, 1]$ the sequence $\omega \in \Omega$ according to $f^i(x) \in \Delta_{\omega_i}$, $i \geqslant 0$, is a bijection between Ω and the points of $[0, 1]$ which are not preimages of zero. Finally, the map ϕ conjugates the map f with the shift σ on Ω. The following properties of the measure μ are easily checked:

(i) $\mu_i \equiv \mu(i) = \mu(\Delta_i) = \rho(\Delta_i) |\Delta_i| = a\mu_0/i^\alpha$, where

$$\rho(\Delta_i) = \frac{(1-a)\,\rho(\Delta_0)}{1 - (i/(i+1))^\alpha}, \qquad i \geqslant 1$$

is the density of the measure in Δ_i, $\mu_0 = 1/(1 + a \sum_{n=1}^{\infty} n^{-\alpha})$, and $|\Delta_i|$ is the diameter of Δ_i.

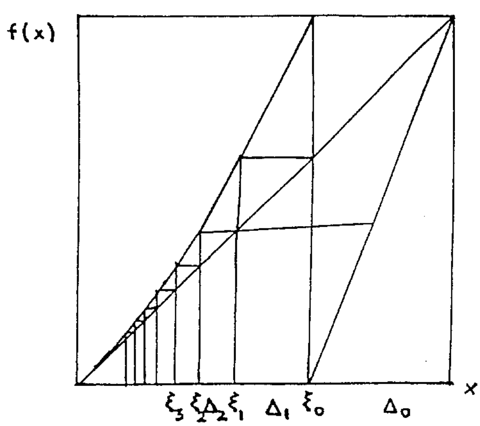

Fig. 1. The Gaspard–Wang map.

(ii) μ is a Markov measure, that is, the stochastic process on Ω, $\omega_n(\omega) = \omega_n$, is a Markov chain with conditional probabilities given by

$$P_{ij} = \mu(\omega_1 = j \mid \omega_0 = i) = \mu(f^{-1}\Delta_j \cap \Delta_i)/\mu(\Delta_i)$$

where we denoted again by μ the measure $\mu \circ \phi^{-1}$.

The corresponding stochastic matrix is easily seen to be

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} \cdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cdots & & \end{pmatrix}$$

with the invariance condition $\sum_{n=0}^{\infty} \mu_n P_{nm} = \mu_m$, which also gives $P_{0n} = (\mu_n - \mu_{n+1})/\mu_0 = \Delta_n$. The structure of this matrix also implies that the chain is irreducible and aperiodic, so that all the states are ergodic[6] and the dynamical system (f, μ) is mixing. We denote by P_n the truncated matrix of conditional probabilities of order n, that is,

$$P_n = \begin{pmatrix} P_{00} & \cdots & P_{0n} \\ 1 & 0 \cdots & 0 \\ \vdots & & \\ 0 & \cdots 1 & 0 \end{pmatrix} \tag{2.1}$$

It is easy to check that all the entries of the matrix P_n^{n+1} are positive, the first line of P_n being the last one of P_n^{n+1}.

We conclude this section with some useful asymptotic expressions; the diameter of each interval scales like

$$|\Delta_n| = |\xi_{n-1} - \xi_n| \underset{n \to \infty}{\sim} \frac{a\alpha}{(1+n)^{\alpha+1}}$$

and the constant slope S_n within each interval behaves like

$$S_n = \frac{|\Delta_{n-1}|}{|\Delta_n|} \underset{n \to \infty}{\sim} 1 + \frac{\alpha+1}{n}, \qquad \Delta_0 = (1-a)$$

The notation $a(n) \sim b(n)$ means $a(n) = b(n) + o(b(n))$, $n \to \infty$.

2.2. Our goal will be to compute the decay of correlations for Hölder continuous functions (of exponent β); without restriction we compute the autocorrelation of Hölder observables g of μ-zero mean, that is,

$$\left| \int g(f^n x)\, g(x)\, d\mu \right| \tag{2.2}$$

It will be clear from the proofs and it was pointed out in ref. 4 that we can enlarge the space of observables g by taking piecewise continuous Hölder functions (with the same exponent or not) on a partition of $[0, 1]$ into finitely many subsegments. We return to this case in the Remark after Lemma 3.2. With abuse of language we call this enlarged space of functions *Hölder functions of exponent* β, where β is the minimum of the local Hölder exponents. We also set $\mathbf{E}(g)$ the expectation of g w.r.t. μ.

We now state our main results.

Theorem 1. Let g be a Hölder function of exponent β with $\mathbf{E}(g)=0$. Then there is a constant $C(g, \alpha, \beta)$ depending only on g, α, and β such that for $\alpha > 3$ and z in the open interval $(1/(\alpha-1), 1/2)$ we have

$$\left| \int g(f^n x)\, g(x)\, d\mu \right| \leqslant C(g, \alpha, \beta) \frac{1}{n^{z\alpha - 1}} \tag{2.3}$$

for $n \geqslant \bar{n}$, where $\bar{n}$ depends on z and $C(g, \alpha, \beta)$. In particular, $\bar{n} \to \infty$ when $z \to 1/2$.

Theorem 1 allows us to prove the existence and the finitness of the following limit, defining the "diffusion coefficient" D_g for the zero mean observable g, at least whenever $\alpha > 4$:

$$D_g = \lim_{n \to \infty} \frac{1}{n} \mathbf{E}\left(\sum_{i=0}^{n-1} g(f^i x) \right)^2 = \mathbf{E}(g)^2 + 2 \sum_{i=1}^{\infty} \mathbf{E}(g(x)\, g(f^i x)) \tag{2.4}$$

We suppose in addition that $D_g \neq 0$ (nondegeneracy of the process); then we can prove the Central Limit Theorem for the observable g:

Theorem 2. Let g satisfy the hypothesis of Theorem 1 and $D_g \neq 0$; if, moreover, we have $\alpha > 30$, then

$$\lim_{n \to \infty} \mu\left(x; \frac{1}{(D_g n)^{1/2}} \sum_{i=0}^{n-1} g(f^i x) < y \right) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{y} e^{-u^2/2}\, du \tag{2.5}$$

3. PROOF OF THEOREM 1: DECAY OF CORRELATIONS

The proof of Theorem 1 consists in several lemmas whose proofs are postponed to the Appendices.

3.1. As usual, in the computation of (2.2), we will shift the description on Ω by using the isomorphism ϕ, that is, we will consider the function $\tilde{g}: \Omega \hookleftarrow$ defined as $\tilde{g}(\omega) = g(\phi^{-1}(\omega))$. We also put $M = \sup_{w \in \Omega} |\tilde{g}(\omega)|$

and for simplicity we set $\tilde{g} = g$. We start by reducing the space Ω, introducing the following two subsets of Ω:

$$\tilde{\Omega}_{n_1} = \{\omega \in \Omega; \omega_0 \cdots \omega_{n_1} \leqslant n_2\} \tag{3.1a}$$

(the symbols $\omega_0 \cdots \omega_{n_1}$ are less than or equal to n_2) and similarly

$$\tilde{\Omega}_{n+n_1} = \{\omega \in \Omega; \omega_0 \cdots \omega_{n+n_1} \leqslant n_2\} \tag{3.1b}$$

where n_1 and n_2 will be chosen later as a function of n; for the moment we simply take $n_2 < n_1 < n$. We identify $\tilde{\Omega}_{n+n_1}$ with the set $\tilde{\Omega}_{n,n_1,n_2}$ defined in the Introduction as a part of the Markov sieve $\mathscr{M}_{n,n_1,n_2}$.

If we generally put $\tilde{\Omega}_m = \{\omega \in \Omega;\ \omega_0 \cdots \omega_m \leqslant n_2\}$, we need to estimate the measure of the complement of $\tilde{\Omega}_m$.

By the invariance of the measure μ it is easily follows that

$$\mu(\mathscr{C}\tilde{\Omega}_m) \leqslant \mu(\mathscr{C}\tilde{\Omega}_{n_2+1}) \left[\frac{m}{n^2+1}\right] \qquad \text{when} \quad m \geqslant n_2 + 1$$

$$\mu(\mathscr{C}\tilde{\Omega}_m) \leqslant \mu(\mathscr{C}\tilde{\Omega}_{n_2+1}) \qquad \text{when} \quad m \leqslant n_2 + 1$$

denoting by $[\cdot]$ the integer part and $\mathscr{C}$ the complementary set.

We are thus led to the estimation of $\mu(\mathscr{C}\tilde{\Omega}_{n_2+1})$. The conditions (3.1) can be viewed as the "rank condition" in the theory of dispersing billiards. As there, the estimation of the complement can be done quite easily with standard probabilistic arguments, but it turns out to be not sufficient for our purposes; we need a more sophisticated analysis (for completeness the preceding standard arguments give a bound like $n_2^{2-\alpha}$).

Lemma 3.1. The following condition holds:

$$\mu(\mathscr{C}\tilde{\Omega}_{n_2+1}) \leqslant \frac{C_1}{n_2^{\alpha-1}} \tag{3.2}$$

where

$$C_1 = \mu_0 a \left[\frac{\alpha}{\alpha - 1} + a\zeta(\alpha)\right]$$

and $\zeta(\alpha)$ is the Riemannian function.

Proof. See Appendix A.

As a consequence, we have

$$\mu(\mathscr{C}\tilde{\Omega}_{n_1}) \leqslant \frac{C_1}{n_2^{\alpha-1}} \left[\frac{n_1}{n_2+1}\right], \qquad \mu(\mathscr{C}\tilde{\Omega}_{n+n_1}) \leqslant \frac{C_1}{n_2^{\alpha-1}} \left[\frac{n+n_1}{n_2+1}\right] \tag{3.3}$$

The fact that the mapping is not uniformly hyperbolic obliges us to approximate the correlation (2.2) with the following one:

$$\int_{\tilde{\Omega}_{n+n_1}} g(\sigma^n \omega)\, g(\omega)\, d\mu(\omega)$$

On the domain $\tilde{\Omega}_{n+n_1}$ we are sufficiently far away from the indifferent fixed point.

The error we incur is easily computed by means of Lemma 3.1:

$$\left| \int_{\Omega} g(\sigma^n \omega)\, g(\omega)\, d\mu - \int_{\tilde{\Omega}_{n+n_1}} g(\sigma^n \omega)\, g(\omega)\, d\mu \right| \leqslant M^2 C_1 \frac{n}{n_2^{\alpha}} \tag{3.4}$$

Warning. Starting from (3.4), we will write all the bounds by using the fact that we will choose later $n_1 = o(n)$ and $n_2 = o(n)$. This implies that only the dominant term will be written; the other are discarded by assuming that they are a given fraction (for example, $c_f = 1/2$) of the dominant term. Collecting all the approximations of this type, we are obliged to choose n greater than a large but finite $\bar{n}$; moreover, all the constants in the following bounds will contain c_f. The value of $\bar{n}$ can be prescribed at the end of the proof once precise relations among n_1, n_2, and α are established (cf. Section 3.2).

We now introduce the cylindrical functions on the cylinders of the form $(\omega_0,..., \omega_{n_1})$, which give a partition of $\tilde{\Omega}_{n_1} \ni \omega$:

$$g_{n_1}(\omega) = \sum_{\omega_0 \cdots \omega_{n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1})\, \chi_{(\omega_0 \cdots \omega_{n_1})}(\omega) \tag{3.5}$$

As a consequence, the indices of summation run over $[0, n_2]$; χ is the characteristic function and

$$g_{n_1}(\omega_0 \cdots \omega_{n_1}) = \frac{1}{\mu(\omega_0 \cdots \omega_{n_1})} \int_{(\omega_0 \cdots \omega_{n_1})} g(\omega)\, d\mu(\omega) \tag{3.6}$$

Using the fact that uniformly for $(\omega_0 \cdots \omega_{n_1}) \subset \tilde{\Omega}_{n_1}$:

$$\begin{aligned} \left| \bigcap_{i=0}^{n_1} f^{-i} \Delta_{\omega_i} \right| &\leqslant \text{const} \times (S_{n_2}^{-1})^{n_1} \\ &\leqslant \text{const} \times \left(1 + \frac{\alpha + 1}{n_2}\right)^{-n_1} \equiv D(n_1, n_2) \end{aligned} \tag{3.7}$$

we get the following result.

Lemma 3.2. The following condition holds:

$$\left| \int_{\tilde{\Omega}_{n+n_1}} g(\sigma^n \omega)\, g(\omega)\, d\mu - \int_{\tilde{\Omega}_{n+n_1}} g_{n_1}(\sigma^n \omega)\, g_{n_1}(\omega)\, d\mu \right| \leqslant 2MD(n_1, n_2)^\beta \quad (3.8)$$

where β is the Hölder exponent.

Proof. See Appendix B.

In order to get a (subexponential) decay we put $n_2 = n_1^\rho$, $0 < \rho < 1$.

Remark. The cylindrical functions are defined on $\tilde{\Omega}_{n_1}$, with length $n_1 + 1$, rather than on $\tilde{\Omega}_{n+n_1}$ since $\tilde{\Omega}_{n+n_1} \subset \sigma^n \tilde{\Omega}_{n+n_1} = \tilde{\Omega}_{n_1}$ and this fact was explicitly used in the proof of Lemma 3.2.

As said in Section 2.2, we can weaken the Hölder property of g by assuming g piecewise Hölder, that is, the Hölder condition is satisfied on each element of a finite partition $\mathscr{P} = \{P_i\}_{i=1}^P$ of $[0, 1]$. This implies that we should neglect the elements of $(\omega_0 \cdots \omega_{n_1}) \subset \tilde{\Omega}_{n_1}$ intersecting the boundaries of $\mathscr{P}$; but we can do that since the measure of each cylinder $(\omega_0 \cdots \omega_{n_1})$ is bounded from above by $n_2[1 + (\alpha + 1)/n_2]^{-n_1}$ (and then decays subexponentially), where the first factor is the asymptotic value of the density $\rho(\Delta_{n_2})$ in the interval Δ_{n_2}. In the particular case that g is already the characteristic function of a Markovian rectangle $(\omega_0 \cdots \omega_m)$, we do not need Lemma 3.2 and we can simply take $n_2 = n_1$, so that the Markov sieve reduces to $\mathscr{M}_{n,n_1}$.

We now try to approximate further the term

$$\int_{\tilde{\Omega}_{n+n_1}} g_{n_1}(\sigma^n \omega)\, g_{n_1}(\omega)\, d\mu(\omega)$$

We call $\hat{\mu}$ the restriction of μ to $\tilde{\Omega}_{n+n_1}$.

By using the representation (3.5) we get

$$\begin{aligned} &\int_{\tilde{\Omega}_{n+n_1}} g_{n_1}(\sigma^n \omega)\, g_{n_1}(\omega)\, d\mu(\omega) \\ &= \sum_{\omega_0 \cdots \omega_{n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1}) \sum_{\omega_n \cdots \omega_{n+n_1}} g_{n_1}(\omega_n \cdots \omega_{n+n_1})\, \hat{\mu}(\omega_0 \cdots \omega_{n_1}; \omega_n \cdots \omega_{n+n_1}) \\ &= \sum_{\omega_0 \cdots \omega_{n+n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1})\, g_{n_1}(\omega_n \cdots \omega_{n+n_1})\, \mu(\omega_0 \cdots \omega_{n+n_1}) \qquad (3.9) \end{aligned}$$

We repeat that all the indices in the last sum are less than or equal to n_2; this condition continues to hold in the following considerations since we will exclusively work on the space $\tilde{\Omega}_{n+n_1}$: all the sums over the indices

$\omega_0, \omega_1, ..., \omega_k$, $k \leqslant n+n_1$, satisfy this condition, and we will simply denote then with $\sum_{\omega_0 \cdots \omega_k}$.[6]

3.2. We now define a new probability measure $\tilde{\mu}$ on $\tilde{\Omega}_{n+n_1}$ by setting

$$\tilde{\mu}(\omega_0 \cdots \omega_m) = \frac{\hat{\mu}(\omega_0 \cdots \omega_m)}{\sum_{(\omega_0 \cdots \omega_{n+n_1})} \mu(\omega_0 \cdots \omega_{n+n_1})}, \qquad m \geqslant 0 \tag{3.10}$$

and extend it to the induced Borel σ-algebra on $\tilde{\Omega}_{n+n_1}$.

Considering on the probability space just constructed $(\tilde{\Omega}_{n+n_1}, \tilde{\mu})$ the stochastic process $\omega_i(\omega) = \omega_i$ (the ith coordinate of $\omega \in \tilde{\Omega}_{n+n_1}$), it is easy to check that it is a nonstationary Markov chain with memory one, as stated by the following lemma, which also holds in the theory of dispersing billiards.[5]

Lemma 3.3. The following condition holds:

$$\forall m > l \geqslant 0, \qquad \tilde{\mu}(\omega_l \cdots \omega_m) = \tilde{\mu}(\omega_l \cdots \omega_p) \prod_{i=p+1}^{m} \tilde{\mu}(\omega_i \mid \omega_{i-1}) \tag{3.11}$$

where

$$\tilde{\mu}(\omega_i \mid \omega_{i-1}) = \mu(\omega_i \mid \omega_{i-1}) \frac{\sum_{\omega_{i+1} \cdots \omega_{n+n_1}} \prod_{j=i+1}^{n+n_1} \mu(\omega_j \mid \omega_{j-1})}{\sum_{\omega_i \cdots \omega_{n+n_1}} \prod_{j=i}^{n+n_1} \mu(\omega_j \mid \omega_{j-1})} \tag{3.12}$$

It is important to realize that in formulas (3.11) and (3.12) the conditional probabilities $\tilde{\mu}(\omega_i \mid \omega_{i-1})$ are computed on the space $\tilde{\Omega}_{n+n_1}$ (and will be successively considered only for $i \leqslant n+n_1$), while $\mu(\omega_i \mid \omega_{i-1})$ are computed on Ω and therefore are stationary.

We now compare the nonstationary conditional probabilities with the stationary ones; we need Lemme 3.1 and the following one:

Lemma 3.4. Let us consider the sum

$$S_n(Z) = \tilde{\sum_{\omega_0 \cdots \omega_n}} \mu(\omega_0 \mid Z)\, \mu(\omega_1 \mid \omega_0) \cdots \mu(\omega_n \mid \omega_{n-1}) \tag{3.13}$$

where $Z \leqslant n_2$ and not all the indices $\omega_0 \cdots \omega_n$ vary simultaneously in the interval $[0, n_2]$ (the tilde distinguishes these from the summation symbols introduced above). Then

$$S_n(Z) \leqslant \frac{1}{\mu_0} \mu(\mathscr{C}\tilde{\Omega}_n) \tag{3.14}$$

[6] By $\sum_{\omega_n \cdots \omega_{n+n_1}} g_{n_1}(\omega_n \cdots \omega_{n+n_1})\, \hat{\mu}(\omega_0 \cdots \omega_{n_1}; \omega_n \cdots \omega_{n+n_1})$, we mean

$$\sum_{\omega_0' \cdots \omega_{n_1}'} g_{n_1}(\omega_0' \cdots \omega_{n_1}')\, \hat{\mu}(\omega_0 \cdots \omega_{n_1}; \omega_n = \omega_0' \cdots \omega_{n+n_1} = \omega_{n_1}')$$

and so on.

Lemma 3.5. The following condition holds:

$$\left|\frac{\tilde{\mu}(\omega_i \mid \omega_{i-1})}{\mu(\omega_i \mid \omega_{i-1})} - 1\right| \leqslant C_2 \frac{n}{n_2^\alpha} \equiv \gamma_n \tag{3.15}$$

The proofs of Lemmas 3.5 and 3.6 are in Appendix C; the constant C_2 can be easily related to μ_0 and the constant C_1 introduced in Lemma 3.1, just looking at Eq. (3.3) and Eq. (C.1). The choice of n_2 in order to get a decay of the upper bound will be discussed at the end of this section.

We now replace in (3.9) the measure μ with the new one; by (3.10) and Eq. (3.3) we immediately get

$$\begin{aligned}\Bigg| &\sum_{\omega_0 \cdots \omega_{n+n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1})\, g_{n_1}(\omega_n \cdots \omega_{n+n_1})\, \mu(\omega_0 \cdots \omega_{n+n_1}) \\ &- \sum_{\omega_0 \cdots \omega_{n+n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1})\, g_{n_1}(\omega_n \cdots \omega_{n+n_1})\, \tilde{\mu}(\omega_0 \cdots \omega_{n+n_1})\Bigg| \leqslant C_3 \frac{n}{n_2^\alpha}\end{aligned} \tag{3.16}$$

where the constant C_3 is the product of M^2 and a term containing C_1.

To treat the second term in the lhs of (3.16), which we call G_n, we now follow the same method as in the proof of Lemma 4.3 in ref. 4. We first add to and subtract from G_n the following expression:

$$\begin{aligned}G_{0,n} = &\sum_{\omega_0 \cdots \omega_{n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1})\, \tilde{\mu}(\omega_0 \cdots \omega_{n_1}) \\ &\times \sum_{\omega_n \cdots \omega_{n+n_1}} g_{n_1}(\omega_n \cdots \omega_{n+n_1})\, \tilde{\mu}(\omega_n \cdots \omega_{n+n_1})\end{aligned} \tag{3.17}$$

which is close to zero, as stated by the following lemma.

Lemma 3.6. The following condition holds:

$$|G_{0,n}| \leqslant C_4 \frac{n}{n_2^\alpha} \tag{3.18}$$

Proof. See Appendix D.

The constant C_4 is the product of M^2 and a term containing C_1 and C_2, as easily follows from the proof.

Therefore we get, using the Markov character of $\tilde{\mu}$,

$$\begin{aligned}|G_n| \leqslant |G_{0,n}| + M^2 &\sum_{\omega_0 \cdots \omega_{n_1}} \tilde{\mu}(\omega_0 \cdots \omega_{n_1}) \\ &\times \sum_{\omega_n} \left| \sum_{\omega_{n_1+1} \cdots \omega_{n-1}} \sum_{i=n_1+1}^{n} \tilde{\mu}(\omega_i \mid \omega_{i-1}) - \tilde{\mu}(\omega_n) \right|\end{aligned} \tag{3.19}$$

We call $G_{1,n}$ the sum over $(\omega_0 \cdots \omega_{n_1})$ in the right-hand side of (3.19): to estimate it we need the following crucial lemma on the uniform rate of mixing on the reduced space $\tilde{\Omega}_{n+n_1}$:

Lemma 3.7 (Rate of mixing). For $\forall \omega_0,\ \omega_{n_2+1} \leqslant n_2$,

$$(P_{n_2}^{n_2+1})_{\omega_0, \omega_{n_2+1}} = \sum_{\omega_1 \cdots \omega_{n_2}} \prod_{i=1}^{n_2+1} \mu(\omega_i \mid \omega_{i-1}) \geqslant \beta_{n_2} \mu(\omega_{n_2+1}) \tag{3.20}$$

where $\beta_{n_2} \sim_{n \to +\infty} 1/n_2$.

Proof. See Appendix E.

Lemma 3.7 allows us to bound subexponentially $G_{1,n}$, as stated by the following lemma, which we will also use in the proof of the CLT:

Lemma 3.8 (Convergence of the nonstationary Markov chain). The following condition holds:

$$|G_{1,n}| \leqslant [1-(1-\gamma_n)^{n_2+1} \beta_{n_2}]^{[(n-n_1)/(n_2+1)]} \chi \tag{3.21}$$

where $\chi \leqslant 4$.

Proof. See Appendix F.

3.3. The leading term to the decay of $|G_n|$ is clearly given by Lemma 3.6 once we prove that the rhs of (3.21) vanishes in the limit $n \to \infty$. In order to get a power-law decay for the bounds given by the inequalities (3.4), (3.16), and (3.18), first set $n_1 = n^\nu$, $0 < \nu < 1$, which, together with $n_2 = n_1^\rho$, gives $n_2 = n_1^\rho = n^{\rho\nu}$; therefore we require $1/\alpha \leqslant \rho\nu$. A more stringent condition of this type comes from the approximation of the nonstationary Markov chain, that is, in order to guarantee the convergence of

$$(1-\gamma_n)^{n_2+1} \sim \left(1 - \text{const}\,\frac{n}{n_2^\alpha}\right)^{n_2}$$

to 1 we must have $1/(\alpha-1) < \rho\nu$.[7] We now return to (3.21). Since β_{n_2} scales as in Lemma 3.7, the rhs of (3.21) decays in a subexponential way whenever $\rho\nu < 1/2$, which joined to the previous lower bound for $\rho\nu$, implies $\alpha > 3$.

In conclusion, for $\alpha > 3$, we get a power law decay of type

$$|\mathbf{E}(g(\sigma^n\omega)\, g(\omega)| \leqslant \text{const} \cdot \frac{1}{n^{z\alpha-1}} \tag{3.22}$$

[7] It is useful to remark that in order to have a subexponential decay of (3.21), it would be sufficient to keep $n - n_1 > n_2 + 1$ or $n_1 = [(n-1)/3]$ instead of $n_1 = n^\nu$ as above (and this evidently does not change the final results). However, the position $n_1 = n^\nu$ will be necessary in the proof of the CLT and we preferred to assume it from the beginning.

where

$$z = \rho v \in \left(\frac{1}{\alpha - 1}; \frac{1}{2}\right)$$

and n is sufficiently large, $n > \bar{n}$, $\bar{n}$ depending on z and the constant entering (3.22), which depends on g, α, and β (see Warning in Section 3). Whenever $\rho v \to 1/2$, the bound (3.21) becomes negligible with respect to n/n_2^α only asymptotically. This concludes the proof of Theorem 1. ∎

4. PROOF OF THEOREM 2: CENTRAL LIMIT THEOREM

The proof of the CLT closely follows that for dispersing billiards quoted in refs. 4 and 5: we want here to emphasize only those parts where new constraints for the parameter α arise. The proof consists in showing that the characteristic function

$$\varphi_n(\lambda) = \mathbf{E}\left\{\exp\left(i \frac{\lambda}{(D_g n)^{1/2}} \sum_{k=0}^{n-1} g(\sigma^k \omega)\right)\right\}$$

converges to $e^{-\lambda^2/2}$, when $n \to \infty$, uniformly in λ on compact sets. A crucial step is to check the Lindeberg condition, which in our case reduces to proving a bound of type

$$\mathbf{E}\left\{\left(\sum_{i=0}^{n} g(\sigma^i \omega)\right)^4\right\} \leqslant \text{const} \cdot n^2 \tag{4.1}$$

The proof of this is performed by first replacing the integral over Ω with the one on $\tilde{\Omega}_{n+n_1}$, as in Section 3.1. Since we have fours sums running over $[0, n]$, the error incur is n^5/n_2^α, which implies $\rho v > 5/\alpha$; but $\rho v < 1/2$ (by Section 3.2), so that we get $\alpha > 10$. Another, more subtle bound on α comes from the following representation for the preceding fourth moment, which is obtained as in (3.9):

$$\begin{aligned}
&\mathbf{E}\left\{\left(\sum_{i=0}^{n} g(\sigma^i \omega)\right)^4\right\} \\
&\quad \sim \sum_{i_1 < i_2 < i_3 < i_4} \sum_{\omega_0 \cdots \omega_{n_1}} \sum_{\omega_{i_1} \cdots \omega_{i_4 + n_1}} g_{n_1}(\omega_{i_1} \cdots \omega_{i_1 + n_1}) \\
&\quad \times g_{n_1}(\omega_{i_2} \cdots \omega_{i_2 + n_1})\, g_{n_1}(\omega_{i_3} \cdots \omega_{i_3 + n_1})\, g_{n_1}(\omega_{i_4} \cdots \omega_{i_4 + n_1}) \\
&\quad \times \mu(\omega_0 \cdots \omega_{n_1}, \omega_{i_1} \cdots \omega_{i_4 + n_1})
\end{aligned}$$

We first consider the terms such that $i_4 - i_1 \leqslant n_1^{1+\psi}$, $0 < \psi < 1$, whose contribution is $n^{3v(1+\psi)+1}$, which in view of (4.1) is smaller than two provided that $(1+\psi)<1/(3\nu)$. The sums over the remaining terms can be split in three parts, namely $i_2-i_1 \geqslant n_1^{1+\psi}$, $i_3-i_2 \geqslant n_1^{1+\psi}$, and $i_4-i_3 \geqslant n_1^{1+\psi}$, and each part can be estimated as in Lemma 3.8 after having suitably added terms close to zero by the zero property of g and Lemma 3.4. In particular we get bound of the type

$$\left(1-\frac{\text{const}}{n_2}\right)^{(i_2-i_1)/n_2} \leqslant \left(1-\frac{\text{const}}{n_2}\right)^{n^{\nu(1+\psi)}/n^{\rho\nu}}$$

which decays provided that $(1+\psi)>2\rho$. Joining this to the previous bound for $1+\psi$, we get $\rho\nu<1/6$, which, together with $\rho\nu>5/\alpha$, finally gives $\alpha>30$.

We now turn to the estimate of the characteristic function $\varphi_n(\lambda)$ by applying Bernstein's method as in refs. 4 and 5.

The first step is to write the sum $\sum_{k=0}^{n-1} g(\sigma^k\omega)$ as

$$\sum_{s=1}^{p} \sum_{k\in\Delta_s(1)} g(\sigma^k\omega)+\sum_{s=1}^{p+1} \sum_{k\in\Delta_s(2)} g(\sigma^k\omega) \tag{4.2}$$

where, for $1\leqslant s\leqslant p$, $|\Delta_s(1)|=[n^{\gamma_1}]$, $|\Delta_s(2)|=[n^{\gamma_2}]$, and they are intercalated; the last one, $\Delta_{p+1}(2)\equiv\hat{\Delta}$, is such that $\hat{\Delta}\leqslant n^{\gamma_1}+n^{\gamma_2}$. Besides $0<\gamma_2<\gamma_1<1$, we set $\tilde{\Delta}=\hat{\Delta}+\Delta_p(2)$, and to simplify the notations we drop the integer part symbol in the next formulas.

Due to this decomposition, (i) we can discard the second term in (4.2), since the limiting distribution of $(D_g n)^{-1/2}\sum_{k=0} g(\sigma^k\omega)$ and $(D_g n)^{-1/2}\sum_{s=1}^{p}\sum_{k\in\Delta_s(1)} g(\sigma^k\omega)$ are the same (see Section 6.2 in ref. 4 for the details); and (ii) all the p stochastic variables $\sum_{k\in\Delta_s(1)} g_{n_1}(\sigma^k\omega)$ [see (4.3)] have the same distribution and this is essential to in applying Lindeberg's condition to (4.4) below.

The analysis of the first term in (4.2) proceeds as for billiards: we first replace g with the cylindrical functions g_{n_1}, making a subexponential error [cf. (3.8)]; therefore we approximate the characteristic function $\varphi_n(\lambda)$ with the following expression, where we introduce the measure $\tilde{\mu}$ according to (3.16):

$$\begin{aligned}
&\sum_{\omega_0\cdots\omega_{n-n^{\gamma_1}-n^{\gamma_2}-\tilde{\Delta}+n_1}} \exp\left\{i\frac{\lambda}{(D_g n)^{1/2}}\sum_{s=1}^{p-1}\sum_{k\in\Delta_s(1)} g_{n_1}(\sigma^k\omega)\right\}\\
&\quad\times\tilde{\mu}(\omega_0\cdots\omega_{n-n^{\gamma_1}-n^{\gamma_2}-\tilde{\Delta}+n_1})\\
&\quad\times\sum_{\omega_{n-n^{\gamma_1}-\tilde{\Delta}}\cdots\omega_{n+n_1}} \exp\left\{i\frac{\lambda}{(D_g n)^{1/2}}\sum_{k\in\Delta_p(1)} g_{n_1}(\sigma^k\omega)\right\}\\
&\quad\times\tilde{\mu}(\omega_{n-n^{\gamma_1}-\tilde{\Delta}}\cdots\omega_{n+n_1}\,|\,\omega_0\cdots\omega_{n-n^{\gamma_1}-n^{\gamma_2}-\tilde{\Delta}+n_1})
\end{aligned} \tag{4.3}$$

By $g_{n_1}(\sigma^k\omega)$ we mean that g_{n_1} is computed on the cylinder with fixed $\omega_k \cdots \omega_{k+n_1}$; this and the choice of the indices in the preceding two sums imply that the sums over k are independent: in particular, we need $n - n^{\gamma_1} - n^{\gamma_2} - \tilde{\Delta} + n_1 < n - n^{\gamma_1} - \tilde{\Delta}$, which implies $\gamma_2 > \nu$.[8]

By invoking Lemma 3.8, we replace the second term in (4.3) with

$$\sum_{\omega_{n-n^{\gamma_1}-\tilde{\Delta}} \cdots \omega_{n+n_1}} \exp\left\{i \frac{\lambda}{(D_g n)^{1/2}} \sum_{k \in \Delta_p(1)} g_{n_1}(\sigma^k\omega)\right\} \tilde{\mu}(\omega_{n-n^{\gamma_1}-\tilde{\Delta}} \cdots \omega_{n+n_1})$$

up to a subexponential error of the type

$$\left(1 - \frac{\text{const}}{n_2}\right)^{n^{\gamma_2}/n_2}$$

which tends to zero whenever $\gamma_2 > 2\rho\nu$, giving $5/\alpha < \rho\nu < 1/2$ or $\alpha > 10$. This bound is already included in the previous one, $\alpha > 30$.

Iterating p times the procedure sketched above, we can factorize the characteristic function in such a way that

$$\left|\varphi_n(\lambda) - \prod_{s=1}^{p} \mathbf{E}\left\{\exp\left(i \frac{\lambda}{(D_g n)^{1/2}} \sum_{k \in \Delta_s(1)} g_{n_1}(\sigma^k\omega)\right)\right\}\right| = o(1) \tag{4.4}$$

when $n \to \infty$ and with the expectation now taken with respect to the original measure μ with a power correction [cf. (3.16)]. The expression (4.4) is bounded independently of λ, so that the convergence to zero is uniform; this plus the Lindeberg condition checked above is sufficient to establish the Central Limit Theorem.

5. NUMERICAL CHECKING

The impossibility of achieving the decay of correlations for $1 < \alpha \leqslant 3$ relies on the inequalities, proved in Section 3,

$$\frac{1}{\alpha - 1} < \rho\nu < \frac{1}{2} \tag{5.1}$$

We could get the desired if, instead of $1/(\alpha - 1) < \rho\nu$, we had $1/\alpha + 1 < \rho\nu$. To this end we should improve Lemma 3.5, bounding γ_n, which is based on Lemma 3.1; alternatively, we could hope to improve the upper bound in (5.1). We show in this section that Lemmas 3.1 and 3.8, on which

[8] This is the point where the comparison between n_1 and n_2 explicitly requires $n_1 = n^\nu$, $\nu \in (0, 1)$ (see also footnote 7).

the upper bound in (5.1) is based, are optimal, in the sense that very accurate numerical computations confirm the analytical bounds. Therefore only a direct improving of Lemma 3.5 could extend Theorem 1 to $\alpha > 1$.

To check Lemma 3.1, we computed numerically the expression $\chi_n = \mu(\tilde{\Omega}_{n+1})$ in (A.1) by means of the recursive relation (A.3), which allows us to go rapidly to order of n up to 8000.

The quantity we are interested in is $E_n \equiv (1-\chi_n) = \mu(\mathscr{C}\tilde{\Omega}_{n+1})$; in Fig. 2 we report $-\log E_n$ versus $\log n$ for different values of $\alpha = 2.5$, 3, 4. In all the cases, the data are fitted with a straight line of slope very close to $\alpha - 1$, thus confirming the theoretical bound given by Lemma 3.1.

In order to check the upper bound in (5.1), we estimate the quantity $G_{1,n}$ (cf. Section 3) in a way different from that used in Lemma 3.8 and better adapted to the numerical computations (it is the same method utilized to prove the subexponential decay of the nonstationary Markov chain for dispersing billiards[5]). The starting point is to subtract from the second term in (3.16) the expression

$$\sum_{\omega_0 \cdots \omega_{n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1})\, \tilde{\mu}(\omega_0 \cdots \omega_{n_1})$$
$$\times \sum_{\omega_n \cdots \omega_{n+n_1}} g_{n_1}(\omega_n \cdots \omega_{n+n_1})\, \tilde{\mu}(\omega_n \cdots \omega_{n+n_1} \mid \bar{\omega}_0 \cdots \bar{\omega}_{n_1})$$

where the symbols $\bar{\omega}_0 \cdots \bar{\omega}_{n_1}$ are fixed and the error is estimated as in Lemma 3.6.

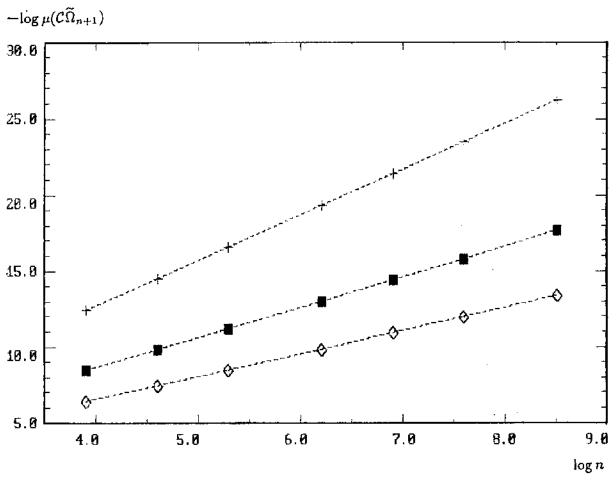

Fig. 2. Graph of $-\log \mu(\mathscr{C}\tilde{\Omega}_{n+1})$ vs. $\log n$ for $\alpha = 2.5$ (diamonds), 3 (squares), 4 (crosses). The slopes are, respectively, 1.523, 2.009, 3.011. The points of intersection with the ordinate axis are functions of α.

The term analogous to $G_{1,n}$, which we call $\tilde{G}_{1,n}$, can now be bounded as

$$|\tilde{G}_{1,n}| \leqslant \sum_{\omega_0 \cdots \omega_{n_1}} \tilde{\mu}(\omega_0 \cdots \omega_{n_1}) \sum_{\omega_n} \Bigg| \sum_{\omega_{n_1+1} \cdots \omega_{n_1}} \tilde{\mu}(\omega_{n_1+1} \mid \omega_{n_1})$$
$$\times \cdots \tilde{\mu}(\omega_n \mid \omega_{n_1}) - \sum_{\omega_{n_1+1} \cdots \omega_{n_1}} \tilde{\mu}(\omega_{n+1} \mid \bar{\omega}_{n_1}) \cdots \tilde{\mu}(\omega_n \mid \omega_{n_1}) \Bigg| \qquad (5.2)$$

We now bound the term in the absolute value in (5.2), which we call R_n:

$$R_n \leqslant \max_{\alpha} \sum_{\omega_{n_1+1} \cdots \omega_{n-1}} \tilde{\mu}(\omega_{n_1+1} \mid \alpha) \cdots \tilde{\mu}(\omega_n \mid \omega_{n-1})$$
$$- \min_{\alpha} \sum_{\omega_{n+1} \cdots \omega_{n-1}} \tilde{\mu}(\omega_{n_1+1} \mid \alpha) \cdots \tilde{\mu}(\omega_n \mid \omega_{n-1})$$

We further define

$$C_{\alpha_1, \omega_n}^{(n-n_1)} = \sum_{\omega_{n_1+1} \cdots \omega_{n-1}} \tilde{\mu}(\omega_{n_1+1} \mid \alpha) \cdots \tilde{\mu}(\omega_n \mid \omega_{n-1})$$
$$M_{\omega_n}^{(n-n_1)} = \max_{\alpha} C_{\alpha_1, \omega_n}^{(n-n_1)}$$
$$m_{\omega_n}^{(n-n_1)} = \min_{\alpha} C_{\alpha_1, \omega_n}^{(n-n_1)}$$

Standard estimates for Markov chains give

$$R_n \leqslant (M_{\omega_n}^{(n-1)} - m_{\omega_n}^{(n-1)})$$
$$\leqslant (M_{\omega_n}^{(n-n_1-n_2+1)} - m_{\omega_n}^{(n-n_1-n_2+1)})$$
$$\times \max_{\alpha,\beta} \Bigg\{ \sum_{\omega_{n_1+n_2+1}}^{+} \Bigg(\sum_{\omega_{n_1+1} \cdots \omega_{n_1+n_2}} \tilde{\mu}(\omega_{n_1+1} \mid \alpha) \cdots \tilde{\mu}(\omega_{n_1+n_2+1} \mid \omega_{n_1+n_2})$$
$$- \sum_{\omega_{n_1+1} \cdots \omega_{n_1+n_2}} \tilde{\mu}(\omega_{n_1+1} \mid \beta) \cdots \tilde{\mu}(\omega_{n_1+n_2+1} \mid \omega_{n_1+n_2}) \Bigg) \Bigg\} \qquad (5.3)$$

The symbol $\sum^{+}$ means that we sum over the $\omega_{n_1+n_2+1}$ for which

$$\sum_{\omega_{n_1+1} \cdots \omega_{n_1+n_2}} \tilde{\mu}(\omega_{n_1+1} \mid \alpha) \cdots \tilde{\mu}(\omega_{n_1+n_2+1} \mid \omega_{n_1+n_2})$$
$$\geqslant \sum_{\omega_{n_1+1} \cdots \omega_{n_1+n_2}} \tilde{\mu}(\omega_{n_1+1} \mid \beta) \cdots \tilde{\mu}(\omega_{n_1+n_2+1} \mid \omega_{n_1+n_2})$$

Using Lemma 3.6, we can replace the nonstationary conditional probabilities with the stationary ones, obtaining for the argument of $\sum^{+}_{\omega_{n_1+n_2+1}}$ in (5.3)

$$\max_{\alpha,\beta} \sum^{+}_{\omega_{n_1+n_2+1}} (\cdot)$$

$$\leqslant \max_{\alpha,\beta} \sum^{+}_{\omega_{n_1+n_2+1}} [(P^{n_2+1}_{n_2})_{\alpha,\omega_{n_1+n_2+1}} - (1-\gamma_n)^{n_2+1} (P^{n_2+1}_{n_2})_{\beta,\omega_{n_1+n_2+1}}]$$

$$\leqslant \left[1-(1-\gamma_n)^{n_2+1} \min_j \frac{\min_i (P^{n_2+1}_{n_2})_{ij}}{\max_i (P^{n_2+1}_{n_2})_{ij}}\right] \tag{5.4}$$

and then we iterate the procedure as in the proof of Lemma 3.8.

The quantity

$$\hat{\beta}_{n_2} = \min_j \frac{\min_i (P^{n_2+1)}_{n_2})_{ij}}{\max_i (P^{n_2+2)}_{n_2})_{ij}}$$

plays the role of β_{n_2} in Lemma 3.8[9]; analytic computations of $\hat{\beta}_{n_2}$ give rough estimates of the type $\hat{\beta}_{n_2} \sim 1/n_2^t$, $t > \alpha + 1$, insufficient to establish

[9] Note, however, that the term in the absolute value in (5.2) must be multiplied by the cardinality of the sum over ω_n (which is n_2), in contrast to the proof of Lemma 3.8. However, this does not affect the subexponential rate of convergence.

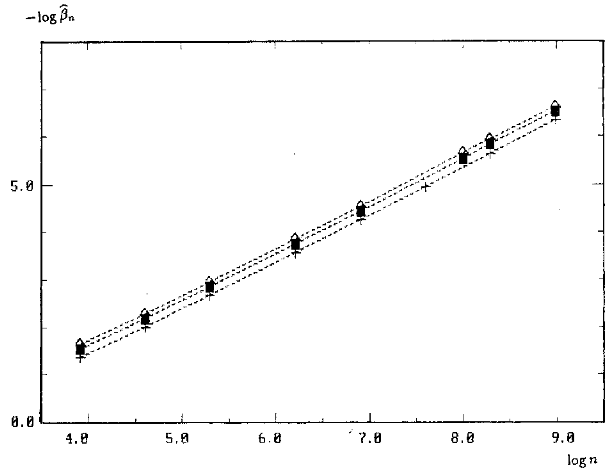

Fig. 3. Graph of $-\log \hat{\beta}_n$ vs. $\log n$ for $\alpha = 2.5$ (diamonds), 3 (squares), 5 (crosses). The slopes are, respectively, 0.997, 0.997, 0.996. The points of intersection with the ordinate axis are functions of α.

Theorem 1. On the other hand, $\hat{\beta}_{n_2}$ can be computed very accurately by using again the recursive relation (A.3).

With the order of n up to 8000 and choosing, for example, $\alpha \in [2.5, 5]$, we find a surprising regularity of the decay, precisely

$$\hat{\beta}_n < \text{const} \cdot \frac{1}{n^t} \tag{5.5}$$

where t is independent of α and almost equal to 0.997 and the constant depends on α.

This is evident, e.g., in Fig. 3, where we report $-\log \hat{\beta}_n$ versus $\log n$ for different values of α; we point out that the statistical analysis on the reduced space $\tilde{\Omega}_{n+n_1}$ showed us the good quantity to study numerically. While a direct numerical computation of (2.2) for nontrivial g is extremely difficult and often unreliable, the mixing rate coefficient expressed by $\hat{\beta}_{n_2}$ can be found with high precision and quality.

APPENDIX A. PROOF OF LEMMA 3.1

We put for simplicity $n_2 = n$ and $P_n = B$. Then we observe that $\mu(\mathscr{C}\tilde{\Omega}_{n+1}) = 1 - \chi_n$, where χ_n is given by

$$\chi_n = \mu(\tilde{\Omega}_{n+1}) = \sum_{i=0}^{n} \mu_i \sum_{j=0}^{n} B_{ij}^{n+1} \tag{A.1}$$

We also define

$$B_{m,i}^{n+1} = L_{n-m,i}, \qquad \forall m = 0,..., n \quad \text{and} \quad \forall i = 0,..., n \tag{A.2}$$

and rename the entries of the first column of the matrix (2.1) as $P_{0i} = P_i$.

It is therefore easy to check the following recursive relation, which is used in the numerical computations of Section 5:

$$L_{m+1,i} = L_{m,0} P_i + L_{m,i+1}, \qquad \forall i = 0,..., n-1 \tag{A.3}$$

which also holds for $i = n$ by setting $L_{m,n+1} = 0$. We will be interested in expressions of the form $\sum_{i=0}^{n} L_{m+1,i}$; repeated use of (A.3) and (A.2) allows us to get the following bounds:

$$\left(\sum_{j \leqslant n} P_j\right)^{m+2} \leqslant \sum_{i=0}^{n} L_{m+1,i} \leqslant \sum_{j \leqslant n} P_j \tag{A.4}$$

We now turn to the computation of $1 - \chi_n$.

Using (A.2), we have

$$1-\chi_n=\mu_0\left(1+a\sum_{n=1}^{\infty}\frac{1}{n^\alpha}-\sum_{j=0}^{n}L_{n,j}-\sum_{i=1}^{n}\frac{a}{i^\alpha}\sum_{j=0}^{n}L_{n-i,j}\right) \tag{A.5}$$

Application of the bounds (A.4) in (A.5) gives, after some manipulations,

$$\frac{1-\chi_n}{\mu_0}\leqslant(n+1)\sum_{j>n}P_j\left(1+a\sum_{i=1}^{n}\frac{1}{i^\alpha}\right)+a\sum_{i>n}\frac{1}{i^\alpha} \tag{A.6}$$

Since

$$\sum_{i=1}^{n}\frac{1}{i^\alpha}\leqslant\sum_{i=1}^{\infty}\frac{1}{i^\alpha}=\zeta(\alpha)<+\infty$$

where $\zeta(\alpha)$ is the Riemann function, we can further bound (A.6) as

$$1-\chi_n\leqslant\mu_0\frac{a}{n^{\alpha-1}}\left[\frac{\alpha}{\alpha-1}+a\zeta(\alpha)\right] \tag{A.7}$$

which completes the proof of the lemma. ∎

APPENDIX B. PROOF OF LEMMA 3.2

The expression on the lhs of (3.8) can be bounded by

$$M[\sup_{\omega\in\tilde{\Omega}_{n+n_1}}|g(\sigma^n\omega)-g_{n_1}(\sigma^n\omega)|+\sup_{\omega\in\tilde{\Omega}_{n+n_1}}|g(\omega)-g_{n_1}(\omega)|] \tag{B.1}$$

Since

$$\begin{aligned}\sup_{\omega\in\tilde{\Omega}_{n+n_1}}|g(\omega)-g_{n_1}(\omega)|&\leqslant\sup_{\omega\in\tilde{\Omega}_{n_1}}|g(\omega)-g_{n_1}(\omega)|\\&\leqslant\sup_{(\omega_0\cdots\omega_{n_1})}\ \sup_{\omega,\omega'\in(\omega_0\cdots\omega_{n_1})}|g(\omega)-g(\omega')|\\&\leqslant\text{const}\cdot\sup_{(\omega_0\cdots\omega_{n_1})}\left|\bigcap_{i=0}^{n_1}f^{-i}\Delta_{\omega_i}\right|^\beta\end{aligned}$$

using (3.7), we get the requested exponential bound for the second term in (B.1). The first term gives a similar bound, remembering that $\sigma^n\tilde{\Omega}_{n+n_1}=\tilde{\Omega}_{n_1}$. ∎

APPENDIX C. PROOFS OF LEMMAS 3.4 AND 3.5

By using the definition of $S_n(Z)$ as given in (3.13) and applaying (3.12), we immediately have

$$\left|\frac{\tilde{\mu}(\omega_i \mid \omega_{i-1})}{\mu(\omega_i \mid \omega_{i-1})} - 1\right| \leqslant \left|\frac{S_{n+n_1-i}(\omega_{i-1}) - S_{n+n_1-i-1}(\omega_i)}{1 - S_{n+n_1-i}(\omega_{i-1})}\right| \tag{C.1}$$

so that Lemma 3.6 is proved once we bound the quantity $S_n(Z)$ as in Lemma 3.5. To prove this, we consider two cases:

(i) Let $Z=0$; then

$$\mu(\omega_0 \mid Z) = P_{0,\omega_0} = \frac{a}{\omega_0^\alpha} - \frac{a}{(\omega_0+1)^\alpha}$$

from which $\mu(\omega_0 \mid Z) \leqslant (1/\mu_0)\, \mu(\omega_0)$. Then $S_n(Z)$ is bounded by

$$S_n(Z) \leqslant \tilde{\sum_{\omega_0 \cdots \omega_n}} \frac{\mu(\omega_0)}{\mu_0} \prod_{i=1}^{n} \mu(\omega_i \mid \omega_{i-1}) \leqslant \frac{1}{\mu_0}\, \mu(\mathscr{C}\tilde{\Omega}_n)$$

(ii) Let $Z \neq 0$; we note that, by hypothesis, $Z \leqslant n_2$; then it must be that $\omega_0 = Z-1$. Therefore

$$S_n(Z) = 1 \cdot \tilde{\sum_{\omega_1 \cdots \omega_n}} \mu(\omega_1 \mid Z-1) \cdots \mu(\omega_n \mid \omega_{n-1})$$

Now if $Z-1=0$, we go back to case (i), otherwise we get as above

$$S_n(Z) = 1 \cdot 1 \cdot \tilde{\sum_{\omega_2 \cdots \omega_n}} \mu(\omega_2 \mid Z-2) \cdots \mu(\omega_n \mid \omega_{n-1})$$

If, continuing the procedure, we are never reduced to case (i) (and this surely happens whenever $n > n_2$), we must get $S_n(Z)=0$, since at least one ω_k, $0 \leqslant k \leqslant n$, must take values larger than n_2. ∎

APPENDIX D. PROOF OF LEMMA 3.6

What we have to estimate is

$$(*) = \sum_{\omega_0 \cdots \omega_{n_1}} g_{n_1}(\omega_0 \cdots \omega_{n_1})\, \tilde{\mu}(\omega_0 \cdots \omega_{n_1})$$

which is very close to zero, with g of zero mean. To make this argument precise, we start by changing $\tilde{\mu}$ to $\hat{\mu}$ in $(*)$. The error is

$$\left|\frac{\tilde{\mu}(\omega_0 \cdots \omega_{n_1})}{\hat{\mu}(\omega_0 \cdots \omega_{n_1})} - 1\right| \leqslant \text{const} \cdot \frac{n}{n_2^\alpha}$$

The next step is to enlarge this sum to the words belonging to $\tilde{\Omega}_{n_1}$, which is the space where we defined the cylindrical functions. The error is clearly bounded by

$$1-\frac{\hat{\mu}(\omega_0\cdots\omega_{n_1})}{\mu(\omega_0\cdots\omega_{n_1})}=\frac{\mu''(\omega_0\cdots\omega_{n_1})}{\mu(\omega_0\cdots\omega_{n_1})}$$

where $\mu(\omega_0\cdots\omega_{n_1})$ is the μ-measure of the word $(\omega_0\cdots\omega_{n_1})\subset\tilde{\Omega}_{n_1}$ and $\mu''(\omega_0\cdots\omega_{n_1})$ is therefore the μ-measure of the word $(\omega_0\cdots\omega_{n_1})\subset\tilde{\Omega}_{n_1}$, for which not all the symbols $\omega_{n_1+1}\cdots\omega_{n+n_1}$ vary simultaneously in the interval $[0, n_2]$. But clearly $\mu''(\omega_0\cdots\omega_{n_1})=\mu(\omega_0\cdots\omega_{n_1})\,S_{n-1}(\omega_{n_1})$. The quantity $S_{n-1}(\omega_{n_1})$ can be estimated as in Eq. (3.14), giving an error const $\cdot\, n/n_2^{\alpha}$. We thus have to bound

$$\sum_{(\omega_0\cdots\omega_{n_1})} g_{n_1}(\omega_0\cdots\omega_{n_1})\,\mu(\omega_0\cdots\omega_{n_1})=\sum_{(\omega_0\cdots\omega_{n_1})\in\tilde{\Omega}_{n_1}}\int_{(\omega_0\cdots\omega_{n_1})} g(\omega)\,d\mu(\omega)$$

But this last expression differs from

$$0=\sum_{(\omega_0\cdots\omega_{n_1})\in\Omega}\int_{(\omega_0\cdots\omega_n)} g(\omega)\,d\mu(\omega)$$

by const $\cdot\,\mu(\mathscr{C}\tilde{\Omega}_{n_1})$ and this concludes the proof. ∎

APPENDIX E. PROOF OF LEMMA 3.7

We have to estimate (3.20), that is

$$(P_{n2}^{n_2+1})_{\omega_0,\omega_{n_2}+1}=\sum_{\omega_1\cdots\omega_{n_2}}\mu(\omega_1\mid\omega_0)\cdots\mu(\omega_{n_2+1}\mid\omega_{n_2}) \tag{E.1}$$

which is surely positive, since the matrix $P_{n2}^{n_2+1}$ is positive, as noted in Section 2. We now consider the last term, that is,

$$(T1)=\sum_{\omega_{n_2-1}}\mu(\omega_{n_2-1}\mid\omega_{n_2-2})\left\{\sum_{\omega_{n_2}}\mu(\omega_{n_2}\mid\omega_{n_2-1})\,\mu(\omega_{n_2+1}\mid\omega_{n_2})\right\}$$

and, for reasons which will be clear in a moment, we want to compare it with

$$(T2)=\mu(\omega_{n_2+1})\sum_{\omega_{n_2-1}}\mu(\omega_{n_2-1}\mid\omega_{n_2-2})\sum_{\omega_{n_2}}\mu(\omega_{n_2}\mid\omega_{n_2-1})$$

In particular, we look for a condition of the type $(T1)\geqslant r(n_2)(T2)$, where $r(n_2)$ is a decreasing function of n_2.

We distinguish two cases:

(i) First, let $\omega_{n_2-2}=b\neq 0$. Then $\omega_{n_2-1}=b-1$ and it is immediate to verify that

$$(T1)=\sum_{\omega_{n_2}}\mu(\omega_{n_2}\mid b-1)\,\mu(\omega_{n_2+1}\mid\omega_{n_2})\geqslant\mu(0\mid 0)\,\mu(\omega_{n_2+1}\mid 0)$$

for any $b>0$.

(ii) Let $\omega_{n_2-2}=b=0$; then

$$(T1)=\sum_{\omega_{n_2-1}}\mu(\omega_{n_2-1}\mid 0)\left\{\sum_{\omega_{n_2}}\mu(\omega_{n_2}\mid\omega_{n_2-1})\,\mu(\omega_{n_2+1}\mid\omega_{n_2})\right\}$$
$$\geqslant\mu(0\mid 0)^2\,\mu(\omega_{n_2+1}\mid 0)$$

In both cases, $(T2)\leqslant\mu(\omega_{n_2+1})$; we will show later that

$$\mu(\omega_{n_2+1}\mid\omega_{n_2}=0)\geqslant\tilde{r}(n_2)\,\mu(\omega_{n_2+1}) \tag{E.2}$$

where $\tilde{r}(n_2)\sim 1/n_2$ for n_2 large. The $r(n_2)$ we are looking for is then given by

$$r(n_2)=\tilde{r}(n_2)\,\mu(0\mid 0)^2\sim\frac{1}{n_2}\qquad(n_2\text{ large}) \tag{E.3}$$

We now prove (E.2), that is, $\mu(\omega_i\mid\omega_{i-1}=0)\geqslant\tilde{r}(n_2)\,\mu(\omega_i)$ for $\omega_i\leqslant n_2$. We first note that by the explicit expressions for the conditional probabilities given in Section 2 we have

$$\mu(\omega_i\mid\omega_{i-1}=0)=\frac{\mu(\omega_i)}{\mu(0)}\left[1-\frac{\mu(\omega_{i+1})}{\mu(\omega_i)}\right]=\frac{\mu(\omega_i)}{\mu(0)}\left[1-\frac{\omega_i^\alpha}{(\omega_{i+1})^\alpha}\right]$$

The function $[1-x^\alpha/(1+x)^\alpha]$ is monotone decreasing to zero and reaches its minimum at $x=n_2$ in our range. Then

$$\tilde{r}(n_2)=\frac{1}{\mu(0)}\left[1-\frac{n_2^\alpha}{(n_2+1)^\alpha}\right]\sim\frac{1}{n_2}\qquad(n_2\text{ large})$$

We now conclude the proof of the lemma. By using the inequality $(T1)\geqslant r(n_2)\,T(2)$ in (E.1) we have

$$(P_{n_2}^{n_2+1})_{\omega_0,\omega_{n_2+1}}\geqslant r(n_2)\,\mu(\omega_{n_2+1})\sum_{\omega_1\cdots\omega_{n_2}}\mu(\omega_1\mid\omega_0)\cdots\mu(\omega_{n_2}\mid\omega_{n_2-1})$$
$$\geqslant\mu(\omega_{n_2+1})\,r(n_2)[1-S_{n_2-1}(\omega_0)]$$

where $S_{n_2-1}(\omega_0)$ is bounded by $[1/\mu(0)]\,\mu(\mathscr{C}\tilde{\Omega}_{n_2-1})$ as in Lemma 3.4. We finish the proof by setting $\beta_{n_2}=r(n_2)[1-S_{n_2-1}(\omega_0)]\sim 1/n_2$ for n_2 large. ∎

APPENDIX F. PROOF OF LEMMA 3.8

We first rewrite $G_{1,n}$ as

$$G_{1,n}=2\sum_{\omega_0\cdots\omega_{n_1}}\tilde{\mu}(\omega_0\cdots\omega_{n_1})\sum_{\omega_n}^{+}\left[\sum_{\omega_{n_1+1}\cdots\omega_{n-1}}\prod_{i=n_1+1}^{n}\tilde{\mu}(\omega_i\,|\,\omega_{i-1})-\tilde{\mu}(\omega_n)\right]$$

where $\sum^+$ means that we sum over all the ω_n for which

$$\sum_{\omega_{n_1+1}\cdots\omega_{n-1}}\prod_{i=n_1+1}^{n}\tilde{\mu}(\omega_i\,|\,\omega_{i-1})\geqslant\tilde{\mu}(\omega_n)$$

We call $G_{2,n}$ the sum over $\sum_{\omega_n}^{+}$: it can be rewritten in the following way (we will divide successively the interval $[n_1, n]$ in multiples of n_2+1 starting from the end):

$$G_{2,n}=\sum_{\omega_n}^{+}\sum_{\omega_{n-n_2-1}}\left[\sum_{\omega_{n_1+1}\cdots\omega_{n-n_2-2}}\prod_{i=n_1+1}^{n-n_2-1}\tilde{\mu}(\omega_i\,|\,\omega_{i-1})-\tilde{\mu}(\omega_{n-n_2-1})\right]$$
$$\times\left[\sum_{\omega_{n-n_2}\cdots\omega_{n-1}}\prod_{i=n-n_2}^{n}\mu\ (\omega_i\,|\,\omega_{i-1})-\tilde{\beta}_{n_2}\tilde{\mu}(\omega_n)\right]$$

where we have subtracted the term

$$\sum_{\omega_n}^{+}\sum_{\omega_{n-n_2-1}}\tilde{\beta}_{n_2}\tilde{\mu}(\omega_n)\left[\sum_{\omega_{n_1+1}\cdots\omega_{n-n_2-2}}\prod_{i=n_1+1}^{n-n_2-1}\tilde{\mu}(\omega_i\,|\,\omega_{i-1})-\tilde{\mu}(\omega_{n-n_2-1})\right]$$

which is zero since $\tilde{\mu}$ is a probability measure on $\tilde{\Omega}_{n+n_1}$; the term $\tilde{\beta}_{n_2}$ will be defined in a moment. Now we replace the conditional probabilities $\tilde{\mu}(\cdot\,|\,\cdot)$ with the corresponding ones $\mu(\cdot\,|\,\cdot)$ in the second square bracket in the expression for $G_{2,n}$; in particular, we get

$$\sum_{\omega_{n-n_2}\cdots\omega_{n-1}}\prod_{i=n-n_2}^{n}\tilde{\mu}(\omega_i\,|\,\omega_{i-1})-\tilde{\beta}_{n_2}\tilde{\mu}(\omega_n)$$
$$\geqslant(1-\gamma_n)^{n_2+1}\,(P_{n_2}^{n_2+1})_{\omega_{n-n_2-1},\omega_n}-\tilde{\beta}_{n_2}\mu(\omega_n)\qquad\text{(F.1)}$$

Now, if we choose $\tilde{\beta}_{n_2}=\beta_{n_2}(1-\gamma_n)^{n_2+1}$, we surely have that the expression in (F.1) is nonnegative by Lemma 3.7. Now we return to $G_{2,n}$; splitting the sum over $n-n_2-1$ into the positive and negative parts $\sum^+$ and $\sum^-$ and by the nonnegativity of (F.1) we have

$$G_{2,n} \leqslant \sum_{\omega_n}^{+} \sum_{\omega_{n-n_2-1}}^{+} \left[\sum_{\omega_{n_1+1} \cdots \omega_{n-n_2-2}} \prod_{i=n_1+1}^{n-n_2-1} \tilde{\mu}(\omega_i \mid \omega_{i-1}) - \tilde{\mu}(\omega_{n-n_2-1}) \right]$$
$$\times \max_{\omega_{n-n_2-1}} \left[\sum_{\omega_{n-n_2} \cdots \omega_{n-1}} \prod_{i=n-n_2}^{n} \tilde{\mu}(\omega_i \mid \omega_{i-1}) - \tilde{\beta}_{n_2} \tilde{\mu}(\omega_n) \right]$$
$$\leqslant [1-(1-\gamma_n)^{n_2+1} \beta_{n_2}]$$
$$\times \sum_{\omega_{n-n_2-1}}^{+} \left[\sum_{\omega_{n_1+1} \cdots \omega_{n-n_2-2}} \prod_{i=n_1+1}^{n-n_2-1} \tilde{\mu}(\omega_i \mid \omega_{i-1}) - \tilde{\mu}(\omega_{n-n_2-1}) \right]$$

We can now iterate the procedure and get

$$|G_{1,n}| \leqslant 2\tilde{\chi}[1-(1-\gamma_n)^{n_2+1} \beta_{n_2}]^{[(n-n_1)/(n_2+1)]}$$

where $\tilde{\chi} < 2$ is the last term of the iteration. ∎

Note Added. After this paper was written, V. Baladi made aware us of the paper by Prellberg and Slawny,[16] where some results for the decay of correlations for 1D mappings with an indifferent fixed point were reported without proof (and without using symbolic dynamics on Markov chains). For our particular map, Mori[17] improved some of our results, by using a technique of Perron–Frobenius type (we thank J. Aaronson for having sent the preprint to us). Another more general and powerful technique of Perron–Frobenius type for transformations of the unit interval with a neutral fixed point has recently been developed in ref. 18.

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